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INERTIAL PLATFORM EQUATIONS (U)

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FOREWORD

This work was done under BuWeps Task Assignment No. RAV32F022/3111/F002-13 003, Job Order No. 6327; BuWeps Task Assignment No. RMGA-36014/3111/F009 03 002, Job Order No. 6333; and BuWeps Task Assignment No. RAV32F021/3111/F002-13 003, Job Order No. 6326. The reports listed in the references have been helpful to the author as introductory material on the subject of inertial navigation. The mathematical relations and equations have been reviewed by John B. Chaney.

SUMMARY

The present paper develops inertial guidance equations for the angular velocities, the linear velocities and the components of the accelerations for nine different platform vertical and azimuth combinations. Platform verticals include geocentric, mass attraction, and geodetic, which are combined with free, north slaved, and grid types of azimuth torquing.

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INTRODUCTION

Inertial navigation employs an inertial input, output system composed of a computer and a gimballed platform containing gyros and accelerometers mounted along the axes of the measurement system. The accelerometers constantly measure acceleration in this system of axes, which is required to have a known orientation with respect to inertial space. The computer is usually referenced to earth's longitude and latitude, or to some other reference frame having known space and time orientation with respect to the measurement axes system. Gravity components are required at all times in both reference axes systems. The following paper describes nine platforms, each representing one possible combination of three azimuth orientation systems (north slaved, free azimuth, grid) and three vertical references (geocentric, mass attraction, plumb line). Mathematical expressions for each platform are derived for the acceleration components, angular rates, and linear and ground velocities.

The most important problem in inertial navigation is the establishment of the level plane through a point above the Earth, which is represented approximately by an Earth reference ellipsoid. Platform level is defined by the choice of the normal or vertical, which may be geocentric, running through the center of the reference ellipsoid or non-geocentric, intercepting the polar axis out of center. The directional coincidence of the normal with the mass attraction is called the mass attraction vertical. The vertical taken normal to the surface of the reference ellipsoid is called the geodetic vertical.

I. GENERALIZED SOLUTION FOR NON-GEOCENTRIC VERTICAL

A. THE COORDINATE SYSTEMS

The right handed coordinate system (x, y, z) is fixed with moving platform. The z axis corresponds to the vertical and is directed downward. The plane $(x\ y)$ defines the plane of a platform. The coordinate system (x', y', z') is parallel to (x, y, z) and origin coincides with center of the Earth. (x_I, y_I, z_I) is the inertial system, fixed in space with respect to a star.

Let $\bar{i}$, $\bar{j}$ and $\bar{k}$ be unit vectors respectively with x, y, z axes or x', y', z'. $\bar{I}$, $\bar{J}$ and $\bar{K}$ are unit vectors along x_I, y_I, z_I axes, respectively.

Geometrical relationship between the coordinate systems is shown in Figure 1.

B. TRANSFORMATION OF THE UNIT VECTORS

Let e_i, m_i, n_i be the directional cosines with corresponding subscripts $i = 1, 2, 3$. Then:

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} e_1, & e_2, & e_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{bmatrix} \begin{bmatrix} I \\ J \\ K \end{bmatrix} \tag{1.1}$$

The moving coordinate system $(x'\ y'\ z')$ can be brought to the system $(x_I\ y_I\ z_I)$ by the three rotations:

(1) The rotation through angle ξ about z' axis brings Ex' in position Ew'.

(2) The rotation through angle $90^\circ + L_k$ about Ew' brings z' into coincidence with z_I.

(3) The rotation through angle $90^\circ - l$ about z_I axis brings x' and y' into coincidence with x_I and y_I, respectively.

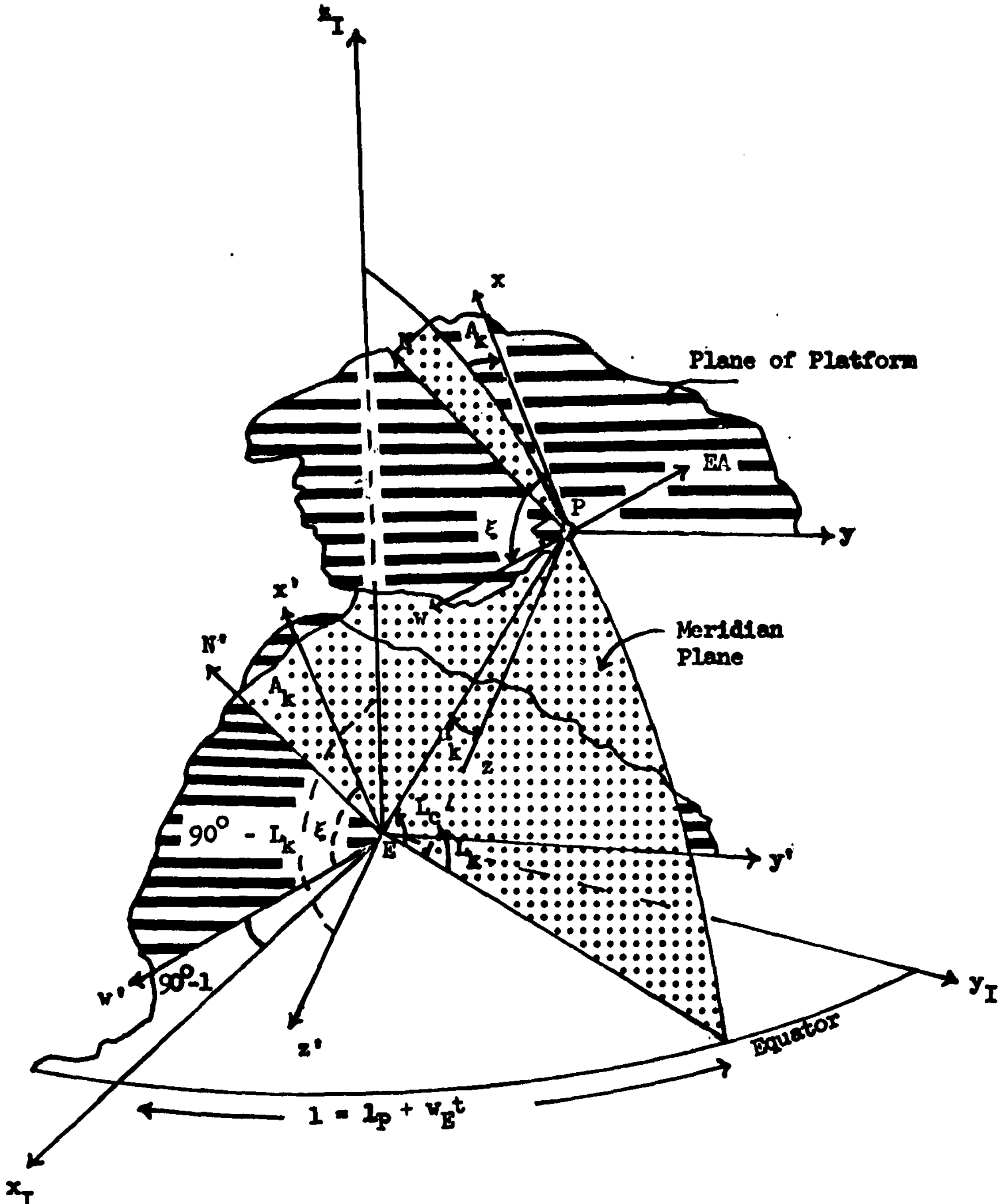

FIGURE 1.

N' || N

w' || w

PN = North direction

PEA= East direction

Pw = West direction

A_k = Azimuth angle of platform

L_c = Geocentric latitude

l_p = Longitude of P

ξ = $90^o + A_k$

L_k = Non-geocentric latitude

k = m , P

m = For mass attraction vertical

P = For plumb line vertical

Finally the direction cosines are:

$$
\begin{aligned}
e_1 &= -\sin A_k \sin l - \cos A_k \sin L_k \cos l \\
e_2 &= \sin A_k \cos l - \cos A_k \sin L_k \sin l \qquad (1.2) \\
e_3 &= \cos A_k \cos L_k
\end{aligned}
$$

$$
\begin{aligned}
m_1 &= -\cos A_k \sin l + \sin A_k \sin L_k \cos l \\
m_2 &= \cos A_k \cos l + \sin A_k \sin L_k \sin l \qquad (1.3) \\
m_3 &= -\sin A_k \cos L_k
\end{aligned}
$$

$$
\begin{aligned}
n_1 &= -\cos L_k \cos l \\
n_2 &= -\cos L_k \sin l \qquad (1.4) \\
n_3 &= -\sin L_k
\end{aligned}
$$

The direction cosines are functions of A_k , L_k and l as seen from the equations (1.2), (1.3) and (1.4).

C. POSITION VECTOR

Let $(\bar{R}_{[E - P]})_I$ be the position vector from the center of Earth to the platform P with respect to inertial space. The components along x , y and z axes can be written for simplicity in the form:

$$[(\bar{R}_{[E - P]})_I]_x = \bar{R}_{EPx}$$

$$[(\bar{R}_{[E - P]})_I]_y = \bar{R}_{EPy}$$

$$[(\bar{R}_{[E - P]})_I]_z = \bar{R}_{EPz}$$

The position vector of a platform P in terms of x , y and z is chosen so that

$$\bar{R}_{EP} = -\bar{i}R_{EPx} - \bar{j}R_{EPy} - \bar{k}R_{EPz} \qquad (1.5)$$

The magnitude of the $\bar{R}_{EP}$ includes radius of the Earth and altitude of the platform:

$$R_{EP} = R + h_c$$

where the subscript c denotes geocentric and R is the radius of the Earth for the point P .

D. ANGULAR VELOCITIES

Let $\bar{w}$ be angular velocity of the moving system P with respect to inertial space. The components of that on x , y and z axes are:

$$\bar{w}_k \ , \ \bar{w}_y \text{ and } \bar{w}_z \text{ , respectively.}$$

The $\bar{w}_E$ is the angular rate vector of the Earth with respect to inertial space and the components along x , y , z are:

$$\bar{w}_{Ex} \ , \ \bar{w}_{Ey} \ , \ \bar{w}_{Ez} \ .$$

The $\bar{w}_P$ is the angular velocity vector of the platform relative to the Earth surface and the components along x , y , z axes are:

$$\bar{w}_{Pk} \ , \ \bar{w}_{Py} \ , \ \bar{w}_{Pz} \ .$$

General angular velocity vector of a platform is:

$$\bar{w} = \bar{w}_E + \bar{w}_P \qquad (1.6)$$

The angular velocity vector of platform P in terms of unit vectors can be expressed:

$$\bar{w} = \bar{i}w_x + \bar{j}w_y + \bar{k}w_z \tag{1.7}$$

The Coriolis law gives:

$$(\dot{\bar{i}})_I = \frac{d}{dt}(\bar{i})_P + \bar{w}x\bar{i}$$

$$\frac{d}{dt}(\bar{i})_P = 0$$

$$\dot{\bar{i}} = \bar{w}x\bar{i} \tag{1.8}$$

From (1.7) and (1.8) after evaluating vector cross products we have:

$$\dot{\bar{i}} = (\bar{i}w_x + \bar{j}w_y + \bar{k}w_z)x\bar{i}$$

or

$$\dot{\bar{i}} = \bar{j}w_z - \bar{k}w_y \tag{1.9}$$

likewise

$$\dot{\bar{j}} = \bar{k}w_x - \bar{i}w_z \tag{1.10}$$

and

$$\dot{\bar{k}} = \bar{i}w_y - \bar{j}w_x \tag{1.11}$$

(1.9) is multiplied by $\bar{k}$, (1.10) is multiplied by $\bar{i}$ and (1.11) is multiplied by $\bar{j}$ and after evaluating cross products we obtain:

$$\begin{aligned} \dot{\bar{j}}x\bar{i} &= \bar{j}w_x \\ \dot{\bar{k}}x\bar{j} &= \bar{k}w_y \\ \dot{\bar{i}}x\bar{k} &= \bar{i}w_z \end{aligned} \tag{1.12}$$

Using the transformation (1.1) we have the vector units of the x , y and z with respect to inertial space:

$$
\begin{aligned}
\bar{I} &= \bar{I}e_1 + \bar{J}e_2 + \bar{K}e_3 \\
\bar{J} &= \bar{I}m_1 + \bar{J}m_2 + \bar{K}m_3 \\
\bar{K} &= \bar{I}n_1 + \bar{J}n_2 + \bar{K}n_3
\end{aligned}
\tag{1.13}
$$

The derivatives of the (1.13) with respect to inertial space are:

$$
\begin{aligned}
\dot{\bar{I}} &= \bar{I}\dot{e}_1 + \bar{J}\dot{e}_2 + \bar{K}\dot{e}_3 \\
\dot{\bar{J}} &= \bar{I}\dot{m}_1 + \bar{J}\dot{m}_2 + \bar{K}\dot{m}_3 \\
\dot{\bar{K}} &= \bar{I}\dot{n}_1 + \bar{J}\dot{n}_2 + \bar{K}\dot{n}_3
\end{aligned}
\tag{1.14}
$$

where $\dot{e}_i = \dfrac{de_i}{dt}$, $\dot{m}_i = \dfrac{dm_i}{dt}$, $\dot{n}_i = \dfrac{dn_i}{dt}$ and $i = 1 , 2 , 3$.

Substitution of (1.13) and (1.14) into the (1.12) gives identities with respect to $\bar{I}$, $\bar{J}$ and $\bar{K}$:

$$
\begin{aligned}
(\bar{I}\dot{m}_1 + \bar{J}\dot{m}_2 + \bar{K}\dot{m}_3) \times (\bar{I}e_1 + \bar{J}e_2 + \bar{K}e_3) &= (\bar{I}n_1 + \bar{J}n_2 + \bar{K}n_3)w_x \\
(\bar{I}\dot{n}_1 + \bar{J}\dot{n}_2 + \bar{K}\dot{n}_3) \times (\bar{I}m_1 + \bar{J}m_2 + \bar{K}m_3) &= (\bar{I}n_1 + \bar{J}n_2 + \bar{K}n_3)w_y \\
(\bar{I}\dot{e}_1 + \bar{J}\dot{e}_2 + \bar{K}\dot{e}_3) \times (\bar{I}n_1 + \bar{J}n_2 + \bar{K}n_3) &= (\bar{I}e_1 + \bar{J}e_2 + \bar{K}e_3)w_z
\end{aligned}
$$

Equating coefficients of either $\bar{I}$, $\bar{J}$ or $\bar{K}$ we obtain:

$$
w_x = \frac{\dot{m}_2 e_3 - \dot{m}_3 e_2}{m_1} \tag{1.15}
$$

$$
w_y = \frac{-\dot{n}_1 m_3 + \dot{n}_3 m_1}{n_2} \tag{1.16}
$$

$$
w_z = \frac{\dot{e}_1 n_2 - \dot{e}_2 n_1}{e_3} \tag{1.17}
$$

Substituting corresponding direction cosines and their derivatives we obtain:

$$v_x = -\dot{L}_k \sin A_k + \dot{l} \cos A_k \cos L_k \qquad (1.18)$$

$$v_y = -\dot{L}_k \cos A_k - \dot{l} \sin A_k \cos L_k \qquad (1.19)$$

$$v_z = \dot{A}_k - \dot{l} \sin L_k \qquad (1.20)$$

where $\dot{l} = \dot{l}_p + w_E$; $\dot{l}_p$ = angular rate of the platform with respect to Earth surface and w_E = inertial Earth rate.

The equations (1.18), (1.19) and (1.20) represent the angular velocity components in general form and applicable to any type of vertical.

E. LINEAR VELOCITIES

The linear velocity of a platform with respect to inertial space is:

$$(\bar{V})_I = \dot{\bar{R}}_{EP} \qquad (1.21)$$

Taking derivative from (1.5) with respect to inertial space we have:

$$\begin{aligned} \dot{\bar{R}}_{EP} = & -\bar{i}\dot{R}_{EPx} - \bar{j}\dot{R}_{EPy} - \bar{k}\dot{R}_{EPz} \\ & - \dot{\bar{i}}R_{EPx} - \dot{\bar{j}}R_{EPy} - \dot{\bar{k}}R_{EPz} \end{aligned} \qquad (1.22)$$

The derivatives of the unit vectors are given by (1.9), (1.10) and (1.11), therefore (1.22) becomes:

$$\begin{aligned} \dot{\bar{R}}_{EP} = & -\bar{i}[-v_z R_{EPy} + v_y R_{EPz} + \dot{R}_{EPx}] \\ & - \bar{j}[-v_x R_{Ez} + v_z R_{EPx} + \dot{R}_{EPy}] \\ & - \bar{k}[-v_y R_{EPx} + v_x R_{EPy} + \dot{R}_{EPz}] \end{aligned} \qquad (1.23)$$

and in terms of x , y and z :

$$[\bar{V}]_I = \bar{i}V_x + \bar{j}V_y + \bar{k}V_z \qquad (1.24)$$

The equations (1.23) and (1.24) give the scalar components of the linear velocity of a platform:

$$V_x = w_z R_{EPy} - w_y R_{EPz} - \dot{R}_{EPx} \tag{1.25}$$

$$V_y = w_x R_{EPz} - w_z R_{EPx} - \dot{R}_{EPy} \tag{1.26}$$

$$V_z = w_y R_{EPx} - w_x R_{EPy} - \dot{R}_{EPz} \tag{1.27}$$

The linear velocity of platform can be split in the components:

$$[\bar{V}]_I = \bar{V}_E + \bar{V}_G \tag{1.28}$$

where $\bar{V}_E$ = Earth's surface velocity with respect to inertial space and

$\bar{V}_G$ = platform velocity with respect to Earth's surface (ground velocity).

The inertial velocity components:

$$\begin{aligned} V_x &= V_{Ex} + V_{Gx} \\ V_y &= V_{Ey} + V_{Gy} \\ V_z &= V_{Ez} + V_{Gz} \qquad \text{where } V_{Ez} = 0 \end{aligned} \tag{1.28a}$$

Analyzing the terms of (1.25), (1.26) and (1.27) we find (See Figure 2-a.)

$$\begin{aligned} R_{EPx} &= R_{EP} \sin \alpha_k \cos A_k \\ R_{EPy} &= -R_{EP} \sin \alpha_k \sin A_k \\ R_{EPz} &= R_{EP} \cos \alpha_k \end{aligned} \tag{1.29}$$

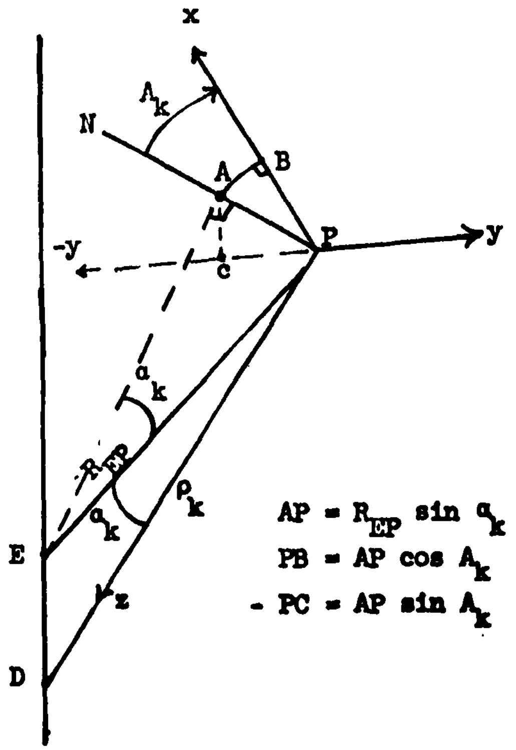

FIGURE 2-a. Components of R_{EP}

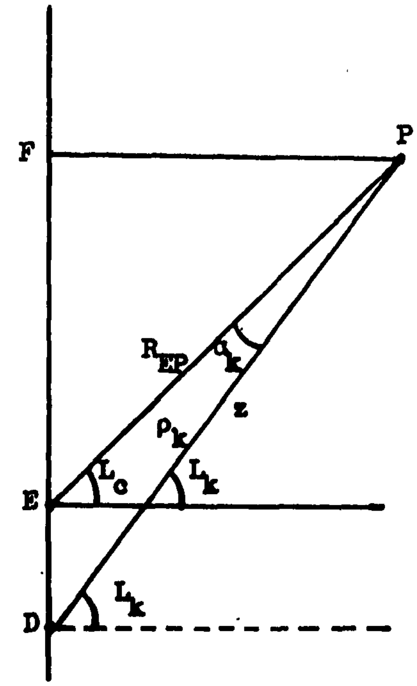

FIGURE 2-b. Meridian plane of P

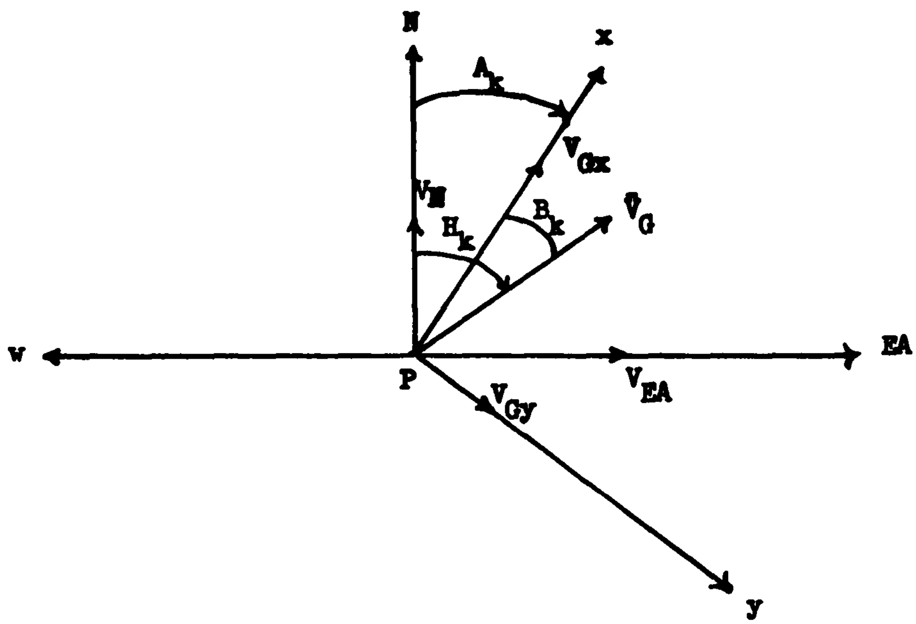

FIGURE 3. Platform horizontal plane

and

$$\dot{R}_{EPx} = \frac{d}{dt}(R_{EP} \sin \alpha_k \cos A_k)$$
$$\dot{R}_{EPy} = -\frac{d}{dt}(R_{EP} \sin \alpha_k \sin A_k) \qquad (1.30)$$
$$\dot{R}_{EPz} = \frac{d}{dt}(R_{EP} \cos \alpha_k)$$

The parameter α_k defines the type of vertical. For geocentric vertical $\alpha_k = 0$. Maximum value of α_k takes place for geodetic vertical: $\alpha_{kmax} = \alpha_P$. From Figure 2-b. we can find:

$$\rho_k = R_{EP} \frac{\cos L_c}{\cos L_k} \qquad (1.31)$$

North ground velocity component is:

$$V_N = \frac{\dot{L}_k R_{EP} \cos L_c}{\cos L_k} \qquad (1.32)$$

and east ground velocity component is:

$$V_{EA} = \dot{l}_P R_{EP} \cos L_c \qquad (1.33)$$

From Figure 3 we can find:

$$V_{Gx} = V_N \cos A_k + V_{EA} \sin A_k$$
$$V_{Gy} = V_{EA} \cos A_k - V_N \sin A_k \qquad (1.34)$$

or

$$V_N = V_{Gx} \cos A_k - V_{Gy} \sin A_k$$
$$V_{EA} = V_{Gx} \sin A_k + V_{Gy} \cos A_k \qquad (1.34a)$$
$$V_{Gz} = -\dot{\rho}_k$$

On basis of the relationships (1.28a) we can express the inertial velocities:

$$V_x = V_{Gx} + \rho_k w_E \cos L_k \sin A_k \tag{1.35}$$

$$V_y = V_{Gy} + \rho_k w_E \cos L_k \cos A_k \tag{1.36}$$

and

$$V_z = V_{Gz} \tag{1.37}$$

F. ACCELERATIONS

The acceleration of platform with respect to inertial space and the mass attraction field can be expressed as follows:

$$\bar{a} = (\ddot{\bar{R}}_{EP})_I + \bar{G}_k \tag{1.38}$$

where $k = c\ ,\ m\ ,\ p\ ;$

c = for geocentric vertical

m = for mass attraction vertical

P = for plumb line vertical

Acceleration vector in terms of the unit vectors of coordinate system $(x\ \ y\ \ z)$ and the scalar components:

$$\bar{a} = \bar{i}a_x + \bar{j}a_y + \bar{k}a_z \tag{1.39}$$

The second derivative of position vector (1.5) with respect to inertial space is

$$\begin{aligned}
(\ddot{\bar{R}}_{EP})_I = {} & \dot{\bar{i}}(-\dot{R}_{EPx} - w_y R_{EPz} + w_z R_{EPy}) \\
& + \dot{\bar{j}}(-\dot{R}_{EPy} - w_z R_{EPx} + w_x R_{EPz}) \\
& + \dot{\bar{k}}(-\dot{R}_{EPz} - w_x R_{EPy} + w_y R_{EPx}) \\
& + \bar{i}[-\ddot{R}_{EPx} + \dot{w}_z R_{EPy} + w_z \dot{R}_{EPy} - \dot{w}_y R_{EPz} - w_y \dot{R}_{EPz}] \\
& + \bar{j}[+\dot{w}_x R_{EPz} + w_x \dot{R}_{EPz} - \dot{w}_z R_{EPx} - w_z \dot{R}_{EPx} - \ddot{R}_{EPy}] \\
& + \bar{k}[\dot{w}_y R_{EPx} + w_y \dot{R}_{EPx} - \dot{w}_x R_{EPy} - w_x \dot{R}_{EPy} - \ddot{R}_{EPz}]
\end{aligned} \tag{1.40}$$

Introducing (1.9), (1.10), (1.11) and (1.40) into (1.38) and comparing with (1.39) we have the components of the acceleration of platform:

$$a_x = -\ddot{R}_{EPx} + (w_y^2 + w_z^2) R_{EPx} + 2 w_z \dot{R}_{EPy} - 2 w_y \dot{R}_{EPz}$$
$$+ \dot{w}_z R_{EPy} - \dot{w}_y R_{EPz} - w_x [w_z R_{EPz} + w_y R_{EPy}] + G_{kx} \tag{1.41}$$

$$a_y = -\ddot{R}_{EPy} + (w_x^2 + w_z^2) R_{EPy} + 2 w_x \dot{R}_{EPz} - 2 w_z \dot{R}_{EPx}$$
$$- \dot{w}_z R_{EPx} + \dot{w}_x R_{EPz} - w_y [w_x R_{EPx} + w_z R_{EPz}] + G_{ky} \tag{1.42}$$

$$a_z = -\ddot{R}_{EPz} + (w_x^2 + w_y^2) R_{EPz} + 2 w_y \dot{R}_{EPx} - 2 w_x \dot{R}_{EPy}$$
$$- \dot{w}_x R_{EPy} + \dot{w}_y R_{EPx} - w_z [w_x R_{EPx} + w_y R_{EPy}] + G_{kz} \tag{1.43}$$

G. RADIUS OF THE EARTH AND MASS ATTRACTION

Assuming the Earth is an ellipsoid we determine the radius of Earth as function of geocentric latitude:

$$R = a\left(1 + \frac{a^2 - b^2}{b^2} \sin^2 L_c\right)^{-\frac{1}{2}} \tag{1.44}$$

where a = equatorial radius,

b = polar radius

Finally

$$R_{EP} = a\left(1 + \frac{a^2 - b^2}{b^2} \sin^2 L_c\right)^{-\frac{1}{2}} + h_c \tag{1.45}$$

Mass attraction vector can be obtained by the gradient of the gravitational potential of the non-rotating Earth ellipsoid:

$$G = \text{grad } V \tag{1.46}$$

where

$$V = \frac{G_E a^2}{R_{EP}} + J \frac{G_E a^4}{R_{EP}^3} \left(\frac{1}{3} - \sin^2 L_c\right) , \tag{1.47}$$

$$G_E = \frac{KM}{a^2} = 979.83123 \text{ gals}$$

and

$$J = .001637$$

The radial component of mass attraction is

$$G_{REP} = -\frac{dV}{dR_{EP}} \tag{1.48}$$

Differentiating gravitational potential (1.47) we have:

$$G_{REP} = \frac{G_E a^2}{R_{EP}^2} + \frac{JG_E a^4}{R_{EP}^4}(1 - 3\sin^2 L_c) \tag{1.49}$$

The normal component to R_{EP} is

$$G_n = \frac{1}{R_{EP}} \frac{dV}{dL_c} \tag{1.50}$$

or

$$G_n = -\frac{2\,JG_E a^4}{R_{EP}^4} \sin L_c \cos L_c \tag{1.51}$$

To give gravity the centrifugal acceleration

$$w_E^2 R_{EP} \cos L_c$$

must be included.

Finally

$$g = w_E^2 R_{EP} \cos L_c + G \tag{1.52}$$

H. LATITUDE CONVERSION

The angle α_m separates mass attraction vertical from geocentric vertical or we might say $\alpha_m = L_m - L_c$.

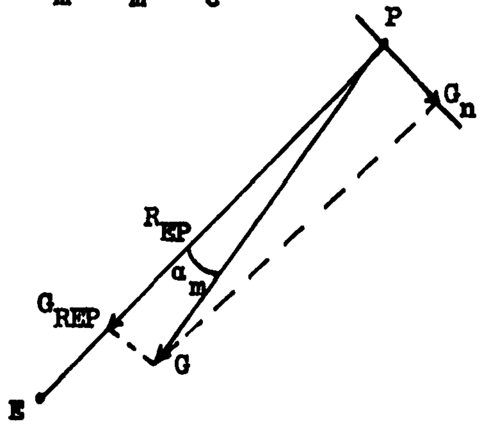

Figure 4.

As it is seen from Figure 4. we have:

$$\tan \alpha_m = \frac{G_n}{G_{R_{EP}}} \tag{1.53}$$

or

$$\tan (L_m - L_c) = \frac{- 2 \, Ja^2 \sin L_c \cos L_c}{R_{EP}^2 + Ja^2(1 - 3 \sin^2 L_c)} \tag{1.54}$$

The equation (1.54) might be used to convert the geocentric latitude to the mass attraction latitude. The solution for the L_m is

$$L_m = L_c + \arctan \left[\frac{- 2 \, Ja^2 \sin L_c \cos L_c}{R_{EP}^2 + Ja^2(1 - 3 \sin^2 L_c)}\right] \tag{1.55}$$

With the meridional section through point P (Figure 5.) we can determine the angle α_P between geocentric vertical and geodetic vertical for point P above the Earth surface.

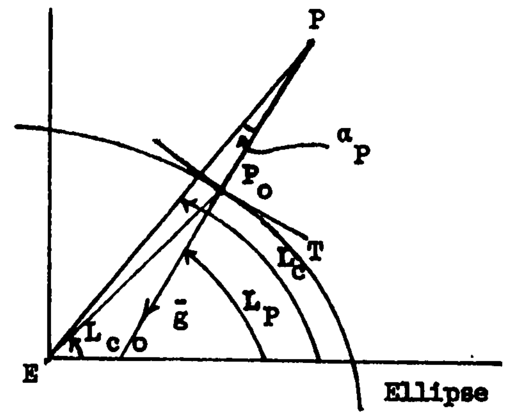

FIGURE 5.

P_oT is level due to $\bar{g}$. $\bar{g}$ is normal to ellipse. The equation of the meridional ellipse through the P_o is

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \tag{1.56}$$

Differentiating (1.56) with respect to u we have

$$\frac{dv}{du} = -\frac{b^2 u}{a^2 v} \tag{1.57}$$

The negative inverse of the slope of the tangent P_oT at given P_o is tan L_P , therefore from (1.57) we have

$$\tan L_P = \frac{a^2 v}{b^2 u} \tag{1.58}$$

but

$$\frac{v}{u} = \tan L_{co} \tag{1.59}$$

From the equations (1.58) and (1.59) we obtain the conversion expression for the geodetic latitude of the point P_o on the surface of the Earth:

$$\tan L_{co} = \frac{\tan L_P}{1 + e_1^2} \tag{1.60}$$

The P is elevated at altitude h_P, therefore L_c differs from L_{co}.

From the triangle EPP_o (Figure 5.) we have

$$\frac{PP_o}{\sin (L_c - L_{co})} = \frac{EP}{\sin (L_P - L_{co})}$$

or

$$\sin (L_c - L_{co}) = \frac{h_P \sin (L_P - L_{co})}{h_P + a(1 + e_1^2 \sin^2 L_{co})^{-\frac{1}{2}}} \tag{1.61}$$

The h_P is indicated by the inertial system. The L_{co} can be computed by the equation (1.60). Solving the equation (1.61) for L_c we find:

$$L_c = L_{co} + \arcsin \left[\frac{h_P \sin (L_P - L_{co})}{h_P + a(1 + e_1^2 \sin^2 L_{co})^{-\frac{1}{2}}}\right] \tag{1.62}$$

where $\quad e_1^2 = \dfrac{a^2 - b^2}{b^2}$;

II. GEOCENTRIC VERTICAL

The consideration that $\alpha_k = 0$ leads to the following results: vertical coincides with radial distance R_{EP} and $L_k = L_c$. The position of a platform is determined by the values of the geocentric latitude L_c, longitude l_p and position vector $\vec{R}_{EP}$. The heading angle A_c determines direction of moving system. The components of radial distance are:

$$R_{EPx} = R_{EPy} = 0 \quad , \quad R_{EPz} = R_{EP}$$

and derivatives become: $\dot{R}_{EPx} = \ddot{R}_{EPx} = \dot{R}_{EPy} = \ddot{R}_{EPy} = 0$.

Applying $L_k = L_c$ to the equations (1.18), (1.19) and (1.20) we have the angular velocities for geocentric vertical:

$$w_x = - \dot{L}_c \sin A_c + \dot{l} \cos A_c \cos L_c \tag{2.1}$$

$$w_y = - \dot{L}_c \cos A_c - \dot{l} \sin A_c \cos L_c \tag{2.2}$$

$$w_z = \dot{A}_c - \dot{l} \sin L_c \tag{2.3}$$

Introducing geocentric vertical conditions into the equations (1.25), (1.26) and (1.27) we have the inertial velocities

$$V_x = - w_y R_{EP} = V_{Gx} + R_{EP} w_E \cos L_c \sin A_c \tag{2.4}$$

$$V_y = \quad w_x R_{EP} = V_{Gy} + R_{EP} w_E \cos L_c \cos A_c \tag{2.5}$$

$$V_z = \quad V_{Gz} \tag{2.6}$$

The ground velocities are:

$$V_N = R_{EP} \dot{L}_c \tag{2.7}$$

I. HEADING ANGLE

The angle between the meridional plane of the P and the plane built by a vertical and the velocity vector and measured in the level plane of the corresponding vertical is the heading angle. The common expression of heading angle is

$$H_k = \text{arc tan}\,[\frac{V_{EA}}{V_N}] \tag{1.63}$$

The heading angle is composed from two angles, the azimuth of platform A_k and bearing angle B_x (Figure 3.):

$$H_k = A_k + B_k \tag{1.64}$$

As we can see from (Figure 3.) the bearing angle is

$$B_k = \text{arc tan}\,[\frac{V_{Gy}}{V_{Gx}}] \tag{1.65}$$

II. GEOCENTRIC VERTICAL

The consideration that $\alpha_k = 0$ leads to the following results: vertical coincides with radial distance R_{EP} and $L_k = L_c$. The position of a platform is determined by the values of the geocentric latitude L_c , longitude l_p and position vector $\overrightarrow{R_{EP}}$. The heading angle A_c determines direction of moving system. The components of radial distance are:

$$R_{EPx} = R_{EPy} = 0 \quad , \quad R_{EPz} = R_{EP}$$

and derivatives become: $\dot{R}_{EPx} = \ddot{R}_{EPx} = \dot{R}_{EPy} = \ddot{R}_{EPy} = 0$.

Applying $L_k = L_c$ to the equations (1.18), (1.19) and (1.20) we have the angular velocities for geocentric vertical:

$$w_x = - \dot{L}_c \sin A_c + \dot{l} \cos A_c \cos L_c \tag{2.1}$$

$$w_y = - \dot{L}_c \cos A_c - \dot{l} \sin A_c \cos L_c \tag{2.2}$$

$$w_z = \dot{A}_c - \dot{l} \sin L_c \tag{2.3}$$

Introducing geocentric vertical conditions into the equations (1.25), (1.26) and (1.27) we have the inertial velocities

$$V_x = - w_y R_{EP} = V_{Gx} + R_{EP} w_E \cos L_c \sin A_c \tag{2.4}$$

$$V_y = \quad w_x R_{EP} = V_{Gy} + R_{EP} w_E \cos L_c \cos A_c \tag{2.5}$$

$$V_z = \quad V_{Gz} \tag{2.6}$$

The ground velocities are:

$$V_N = R_{EP} \dot{L}_c \tag{2.7}$$

$$V_{EA} = R_{EP}\,\dot{i}_P \cos L_c \tag{2.8}$$

then

$$V_{Gx} = V_N \cos A_c + V_{EA} \sin A_c \tag{2.9}$$

$$V_{Gy} = V_N \sin A_c - V_{EA} \cos A_c \tag{2.9a}$$

and

$$V_{Gz} = - \dot{R}_{EP} \tag{2.10}$$

Observing all the conditions for geocentric vertical we obtain the simplified acceleration components from the equations (1.41), (1.42) and (1.43):

$$a_x = - \dot{w}_y R_{EP} - w_x w_z R_{EP} - 2\, w_y \dot{R}_{EP} + G_x \tag{2.11}$$

$$a_y = \dot{w}_x R_{EP} - w_y w_z R_{EP} + 2\, w_x \dot{R}_{EP} + G_y \tag{2.12}$$

$$a_z = (w_x^2 + w_y^2)\, R_{EP} - \dot{R}_{EP} + G_z \tag{2.13}$$

A. NORTH SLAVED PLATFORM

Assuming the azimuth angle is zero

$$A_c = 0$$

we slave the platform to the north. The angular velocities become:

$$w_x = \dot{i} \cos L_c \tag{2.14}$$

$$w_y = - \dot{L}_c \tag{2.15}$$

$$w_z = - \dot{i} \sin L_c \tag{2.16}$$

Introducing $\dot{i} = \dot{i}_P + w_E$ and the ground velocities from (2.7) and (2.8) into the equations (2.14), (2.15) and (2.16) we have:

$$w_x = \frac{V_{EA}}{R_{EP}} + w_E \cos L_c \qquad (2.17)$$

$$w_y = - \frac{V_N}{R_{EP}} \qquad (2.18)$$

$$w_z = - \left(\frac{V_{EA}}{R_{EP}} + w_E \cos L_c\right) \tan L_c \qquad (2.19)$$

To keep the platform slaved to north and in level with respect to geocentric vertical the gyros should be torqued at the following rates:

The north gyro:

$$t_N = - \frac{V_{EA}}{R_{EP}} - w_E \cos L_c \qquad (2.20)$$

The east gyro:

$$t_{EA} = \frac{V_N}{R_{EP}} \qquad (2.21)$$

The vertical gyro:

$$t_V = \frac{V_{EA}}{R_{EP}} \tan L_c + w_E \sin L_c \qquad (2.22)$$

The linear velocities can be expressed in the form:

$$V_x = V_N \qquad (2.23)$$

$$V_y = V_{EA} + R_{EP} w_E \cos L_c \qquad (2.24)$$

$$V_z = V_{Gz} = - \dot{R}_{EP} \qquad (2.25)$$

The latitude of position can be computed from equation (2.23), this gives:

$$L_c = \int_{t_o}^{t_1} \frac{V_N}{R_{EP}} \, dt \tag{2.26}$$

Solving the equation (2.8) for $\dot{l}_p$ and integrating as time function we obtain the longitude of position:

$$l_p = \int_{t_o}^{t_1} \frac{V_{EA}}{R_{EP} \cos L_c} \, dt \tag{2.27}$$

The altitude h_c of the platform can be computed from the equation (2.25):

$$R_{EP} = - \int_{t_o}^{t_1} V_{Gz} \, dt$$

and

$$h_c = R_{EP} - R \tag{2.28}$$

Analyzing the expressions of acceleration (2.11), (2.12) and (2.13) due to north slaved condition we have:

$$a_x = \dot{V}_N - \frac{V_{Gz} V_N}{R_{EP}} + \frac{V_{EA}^2}{R_{EP}} \tan L_c + 2 \, V_{EA} w_E \sin L_c$$

$$+ R_{EP} w_E^2 \sin L_c \cos L_c + G_x \tag{2.29}$$

$$a_y = \dot{V}_{EA} - V_{Gz} \left[\frac{V_{EA}}{R_{EP}} + 2 \, w_E \cos L_c\right] - \frac{V_N V_{EA}}{R_{EP}} \tan L_c$$

$$- 2 \, V_N w_E \sin L_c \tag{2.30}$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_N^2 + V_{EA}^2}{R_{EP}} + 2\,V_{EA} w_E \cos L_c$$

$$+ R_{EP} w_E^2 \cos^2 L_c + G_z \qquad (2.31)$$

We can introduce $g_x = R_{EP} w_E^2 \sin L_c \cos L_c + G_x$

and $g_z = R_{EP} w_E^2 \cos^2 L_c + G_z$

From the equations (2.29), (2.30) and (2.31) we can find:

V_N , V_{EA} and V_{Gz} . The component a_x is sensed by north accelerometer. The accelerometer should be torqued with the following signal:

$$A_{ccx} = -\,(a_x - \dot{V}_N) \qquad (2.32)$$

The a_y is sensed by east accelerometer. The accelerometer should be torqued at the rate:

$$A_{ccy} = -\,(a_y - \dot{V}_{EA}) \qquad (2.33)$$

The vertical acceleration a_z is sensed by vertical accelerometer. The torquing rate is:

$$A_{ccz} = -\,(a_z - \dot{V}_{Gz}) \qquad (2.34)$$

B. FREE AZIMUTH PLATFORM

No angular rate exists for azimuth free platform along z axis, therefore

$$v_z = 0$$

From the equation (2.3):

$$\dot{A}_c = (\dot{i}_P + w_E) \sin L_c \tag{2.35}$$

Integrating the equation (2.35) we find the expression to compute azimuth angle:

$$A_c = \int_{t_o}^{t_1} (\dot{i}_P + w_E) \sin L_c \, dt \tag{2.36}$$

Other two angular velocities along x and y axes are, respectively:

$$v_x = - \dot{L}_c \sin A_c + \dot{i}_P \cos A_c \cos L_c + w_E \cos A_c \cos L_c \tag{2.37}$$

and

$$v_y = - \dot{L}_c \cos A_c - \dot{i}_P \cos L_c \sin A_c - w_E \sin A_c \cos L_c \tag{2.38}$$

or in the terms of ground velocity:

$$v_x = - \frac{V_N}{R_{EP}} \sin A_c + \frac{V_{EA}}{R_{EP}} \cos A_c + w_E \cos A_c \cos L_c$$

$$= \frac{V_{Gy}}{R_{EP}} + w_E \cos A_c \cos L_c \tag{2.39}$$

and

$$v_y = - \frac{V_{Gx}}{R_{EP}} - w_E \sin A_c \cos L_c \tag{2.40}$$

The torquing rates of x axis gyro and y axis gyro to keep the platform in level are, respectively:

$$t_x = -\frac{V_{Gy}}{R_{EP}} - w_E \cos A_c \cos L_c \tag{2.41}$$

$$t_y = \frac{V_{Gx}}{R_{EP}} + w_E \sin A_c \cos L_c \tag{2.42}$$

No torquing rates are applied to azimuth axis - z . The x axis can point any place at any time during flight. Knowing initial azimuth angle of platform we can find deviation from north at a later time by the known rate $(\dot{l}_p + w_E) \sin L_c$ in the equation (2.35).

The ground velocities V_{Gx} , V_{Gy} , and V_{Gz} keep the same form of the expressions, therefore, the linear velocities for the azimuth free platform are:

$$V_x = V_{Gx} + R_{EP} w_E \cos L_c \sin A_c \tag{2.43}$$

$$V_y = V_{Gy} + R_{EP} w_E \cos L_c \cos A_c \tag{2.44}$$

$$V_z = V_{Gz} \tag{2.45}$$

Applying generalized expressions of acceleration to a platform oriented free in azimuth we have the sensed accelerations along axes of the moving system:

$$a_x = \dot{V}_{Gx} - V_{Gz} \left[\frac{V_{Gx}}{R_{EP}} + 2 w_E \cos L_c \sin A_c\right] + V_{Gy} w_E \sin L_c$$

$$+ R_{EP} w_E^2 \sin L_c \cos L_c \cos A_c + G_x \tag{2.46}$$

$$a_y = \dot{V}_{Gy} - V_{Gz} \left[\frac{V_{Gy}}{R_{EP}} + 2\, w_E \cos L_c \cos A_c\right] - V_{Gx} w_E \sin L_c$$

$$- R_{EP} w_E^2 \sin L_c \cos L_c \sin A_c + G_y \tag{2.47}$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_{Gx}^2 + V_{Gy}^2}{R_{EP}} + 2\, w_E \left[V_{Gx} \cos A_c - V_{Gy} \sin A_c\right] \cos^2 L_c$$

$$+ R_{EP} w_E^2 \cos^2 L_c + G_z \tag{2.48}$$

The three accelerometers should be torqued by the following torquing rates:

$$A_{ccx} = - (a_x - \dot{V}_{Gx}) \tag{2.49}$$

$$A_{ccy} = - (a_x - \dot{V}_{Gy}) \tag{2.50}$$

and

$$A_{ccz} = - (a_z - \dot{V}_{Gz}) \tag{2.51}$$

C. GRID PLATFORM

The x axis of the platform makes a constant angle with a particular initial meridian at any time during flight such that the x axis of the platform stays in parallel planes normal to the equator during flight. This can happen when the earth rate is applied to torque the azimuth axis.

The angular velocity along z axis is

$$w_z = - w_E \sin L_c \tag{2.52}$$

From this we have:

$$\dot{A}_c = \dot{i}_P \sin L_c \qquad (2.53)$$

The x axis deviates from north direction by angle A_c, which is the time integral of the angular rate of the platform with respect to the earth surface; therefore,

$$A_c = \int_{t_o}^{t_1} \dot{i}_P \sin L_c \, dt \qquad (2.54)$$

In this case we have the following angular velocities in the terms of ground velocities:

$$w_x = \frac{V_{Gy}}{R_{EP}} + w_E \cos L_c \cos A_c \qquad (2.55)$$

$$w_y = - \frac{V_{Gx}}{R_{EP}} - w_E \cos L_c \sin A_c \qquad (2.56)$$

and

$$w_z = - w_E \sin L_c \qquad (2.57)$$

The torquing rates of the gyros:

$$t_x = -v_x \tag{2.58}$$

$$t_y = -v_y \tag{2.59}$$

and

$$t_z = -v_z \tag{2.60}$$

These rates keep the platform in level and satisfy the grid condition.

The equations (2.4) to (2.6) are applied for the grid type platform.

The ground velocities are the same as in the case of free azimuth platform, therefore:

$$V_{Gx} = R_{EP}(\dot{L}_c \cos A_c + \dot{l}_P \cos L_c \sin A_c) \tag{2.61}$$

$$V_{Gy} = -R_{EP}(-\dot{L}_c \sin A_c + \dot{l}_P \cos L_c \cos A_c) \tag{2.62}$$

and

$$V_{Gz} = -\dot{R}_{EP} \tag{2.63}$$

From these equations we have:

$$L_c = \int_{t_o}^{t_1} \frac{V_{Gx} \cos A_c - V_{Gy} \sin A_c}{R_{EP}} dt \tag{2.64}$$

$$l_P = \int_{t_o}^{t_1} [\frac{V_{Gx} \sin A_c + V_{Gy} \cos A_c}{R_{EP} \cos L_c}]dt \tag{2.65}$$

and

$$h_c = R_{EP} - R \tag{2.66}$$

The position is determined by the expressions (2.64), (2.65) and (2.66). The ground velocities are computed from the acceleration equations; those we obtain introducing the conditions of the grid platform into the equations (2.11), (2.12) and (2.13):

$$a_x = \dot{V}_{Gx} - V_{Gz}\left[\frac{V_{Gx}}{R_{EP}} + 2\,w_E \cos L_c \sin A_c\right]$$

$$+ 2\,w_E V_{Gy} \sin L_c + R_{EP} w_E^2 \sin L_c \cos L_c \cos A_c + G_x \tag{2.67}$$

$$a_y = \dot{V}_{Gy} - V_{Gz}\left[\frac{V_{Gy}}{R_{EP}} + 2\,w_E \cos L_c \cos A_c\right]$$

$$- 2\,w_E V_{Gx} \sin L_c - R_{EP} w_E^2 \sin L_c \cos L_c \sin A_c + G_y \tag{2.68}$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_{Gx}^2 + V_{Gy}^2}{R_{EP}} + 2[V_{Gx} \cos A_c - V_{Gy} \sin A_c] w_E \cos^2 L_c$$

$$+ R_{EP} w_E^2 \cos^2 L_c + G_z \tag{2.69}$$

The last two terms of each equation are components of the gravity.

The torquing signals of the accelerometers along x , y and z axes:

$$A_{ccx} = -(a_x - \dot{V}_{Gx}) \tag{2.70}$$

$$A_{ccy} = -(a_y - \dot{V}_{Gy}) \tag{2.71}$$

and

$$A_{ccz} = -(a_z - \dot{V}_{Gz}) \tag{2.72}$$

III. MASS ATTRACTION VERTICAL

Since the z axis coincides with the mass attraction vertical the components of mass attraction acceleration become

$$G_x = 0$$

$$G_y = 0$$

and

$$G_z = G$$

The angle between geocentric vertical and mass attraction vertical is

$$\alpha_k = \alpha_m$$

The latitude becomes

$$L_k = L_m$$

and heading angle

$$A_k = A_m$$

Taking into account all those conditions we might use generalized solutions of the non-geocentric vertical to analyze any orientation of a platform with respect to the north direction.

Applying the inertial velocities for mass attraction vertical from (1.35) to (1.37) we have

$$V_x = V_{Gx} + \rho_m w_E \cos L_m \sin A_m$$

$$V_y = V_{Gy} + \rho_m w_E \cos L_m \cos A_m \qquad (3.1)$$

$$V_z = V_{Gz}$$

The inertial accelerations in the form of (1.41) to (1.43) might be interpreted in the terms of the inertial velocities and angular velocities as follows

$$a_x = \dot{V}_x + w_y V_z - w_z V_y$$

$$a_y = \dot{V}_y + w_z V_x - w_x V_z \tag{3.2}$$

and

$$a_z = \dot{V}_z + w_x V_y - w_y V_x + G$$

Introducing the quantities of the set (3.1) into the equations (3.2), the inertial accelerations sensed by the accelerometers become:

$$a_x = \dot{V}_{Gx} - V_{Gy}\,[w_z - w_E \sin L_m] + V_{Gz}\,[w_y - w_E \cos L_m \sin A_m]$$

$$+ \;\rho_m w_E^2 \sin L_m \cos L_m \cos A_m \tag{3.3}$$

$$a_y = \dot{V}_{Gy} + V_{Gx}\,[w_z - w_E \sin L_m] - V_{Gz}\,[w_x + w_E \cos L_m \cos A_m]$$

$$- \;\rho_m w_E^2 \sin L_m \cos L_m \sin A_m \tag{3.4}$$

and

$$a_z = \dot{V}_{Gz} - V_{Gx}\,[w_y - w_E \cos L_m \sin A_m] + V_{Gy}\,[w_x + w_E \cos L_m \operatorname{Cos} A_m]$$

$$+ \;\rho_m w_E^2 \cos^2 L_m + G \tag{3.5}$$

The solution of the differential equations (3.3) to (3.5) gives the ground velocities

$$V_{Gx}\,,\quad V_{Gy}\,,\quad \text{and}\quad V_{Gz}\,.$$

The position information is obtainable from the ground velocities, which are time integrals.

The angular velocities due to mass attraction vertical:

$$w_x = -\dot{L}_m \sin A_m + \dot{i} \cos A_m \cos L_m \tag{3.6}$$

$$w_y = -\dot{L}_m \cos A_m - \dot{i} \cos L_m \sin A_m \tag{3.7}$$

and

$$w_z = \dot{A}_m - \dot{i} \sin L_m \tag{3.8}$$

Recalling (1.31) for mass attraction vertical we have

$$\rho_m = R_{EP} \frac{\cos L_c}{\cos L_m} \tag{3.9}$$

and

$$L_c = L_{co} + \text{arc sin} \left[\frac{h_m \sin (L_m - L_{co})}{h_m + a(1 + e_1^2 \sin^2 L_{co})^{-\frac{1}{2}}}\right] \tag{3.10}$$

that is identical with the expression (1.62). To convert latitudes L_m and L_c we can use the formula (1.55).

A. NORTH SLAVED PLATFORM

At any time during motion the x axis is not separated from the north direction; azimuth angle A_m is zero. The angular rates of the moving system become:

$$w_x = \frac{V_{EA}}{\rho_m} + w_E \cos L_m \tag{3.11}$$

$$w_y = -\frac{V_N}{\rho_m} \tag{3.12}$$

and

$$w_z = -\frac{V_{EA}}{\rho_m} \tan L_m - w_E \sin L_m \tag{3.13}$$

To keep the platform in level and in proper verticality the gyros are torqued by the following rates:

$$t_x = - w_x$$

$$t_y = - w_y \tag{3.14}$$

and

$$t_z = - w_z$$

The inertial velocities due to the mass attraction vertical and $A_m = 0$ from the equations (1.35) to (1.37) might be expressed in the form:

$$V_x = V_N$$

$$V_y = V_{EA} + \rho_m w_E \cos L_m \tag{3.15}$$

$$V_z = V_{Gz}$$

where

$$V_N = \rho_m \dot{L}_m$$

$$V_{EA} = \rho_m \dot{l}_P \cos L_m \tag{3.16}$$

and

$$V_{Gz} = - \dot{\rho}_m$$

The inertial accelerations of the north slaved platform become:

$$a_x = \dot{V}_N - \frac{V_N V_{Gz}}{\rho_m} + \frac{V_{EA}^2}{\rho_m} \tan L_m$$

$$+ 2 V_{EA} w_E \sin L_m + \rho_m w_E^2 \sin L_m \cos L_m \tag{3.17}$$

$$a_y = \dot{V}_{EA} - V_{Gz} \left[\frac{V_{EA}}{\rho_m} + 2\, w_E \cos L_m\right] - 2\, V_N w_E \sin L_m - \frac{V_N V_{EA}}{\rho_m} \tan L_m \qquad (3.18)$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_N^2 + V_{EA}^2}{\rho_m} + 2\, V_{EA} w_E \cos L_m + \rho_m w_E^2 \cos^2 L_m + G \qquad (3.19)$$

The torquing rates of the accelerometers are

$$A_{ccx} = -\,(a_x - \dot{V}_N) \qquad (3.20)$$

$$A_{ccy} = -\,(a_y - \dot{V}_{EA}) \qquad (3.21)$$

$$A_{ccz} = -\,(a_z - \dot{V}_{Gz}) \qquad (3.22)$$

The position coordinates are determined by the set (3.16):

$$L_m = \int_{t_o}^{t_1} \frac{V_N}{\rho_m}\, dt \qquad (3.23)$$

$$l_p = \int_{t_o}^{t_1} \frac{V_{EA}}{\rho_m \cos L_m}\, dt \qquad (3.24)$$

and

$$h_m = -\int_{t_o}^{t_1} V_{Gz}\,dt - \frac{a(1 + e_1^2 \sin^2 L_{co})^{-\frac{1}{2}} \cos L_{co}}{\cos L_m} \qquad (3.25)$$

where L_{co} and L_m are related by the equations (1.55) and (3.10). The heading angle is computed by

$$H_m = \arctan \left[\frac{V_{EA}}{V_N}\right] \qquad (3.26)$$

B. FREE AZIMUTH PLATFORM

This case requires

$$w_z = 0 \qquad (3.26)$$

Other two inertial angular velocities are

$$w_x = \frac{V_{Gy}}{\rho_m} + w_E \cos L_m \cos A_m \qquad (3.27)$$

and

$$w_y = -\frac{V_{Gx}}{m} - w_E \cos L_m \sin A_m \qquad (3.28)$$

From here the platform alignment rates are as follows:

$$t_x = -w_x$$

$$t_y = -w_y \qquad (3.29)$$

and

$$t_z = 0$$

No torquing rates exist along azimuth axis. The platform deviation angle from the north direction is determined by

$$A_m = \int_{t_o}^{t_1} \left[\frac{V_{GEA}}{\rho_m} \tan L_m + w_E \sin L_m\right] dt \tag{3.30}$$

The inertial velocities hold the relationships of the set (3.1).

Supplying the azimuth free condition to the general expressions of the inertial accelerations (3.3) to (3.5) we obtain

$$a_x = \dot{V}_{Gx} - V_{Gz}\left[\frac{V_{Gx}}{\rho_m} + 2\, w_E \cos L_m \sin A_m\right]$$

$$+ w_E V_{Gy} \sin L_m + \rho_m w_E^2 \sin L_m \cos L_m \cos A_m \tag{3.31}$$

$$a_y = \dot{V}_{Gy} - V_{Gz}\left[\frac{V_{Gy}}{\rho_m} + 2\, w_E \cos L_m \cos A_m\right]$$

$$- w_E V_{Gx} \sin L_m - \rho_m w_E^2 \sin L_m \cos L_m \sin A_m \tag{3.32}$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_{Gx}^2 + V_{Gy}^2}{\rho_m} + 2\, w_E V_{Gx} \cos L_m \sin A_m$$

$$+ 2\, w_E V_{Gy} \cos L_m \cos A_m + \rho_m w_E^2 \cos^2 L_m + G \tag{3.33}$$

The accelerometers are torqued by the rates:

$$A_{ccx} = -(a_x - \dot{V}_{Gx})$$

$$A_{ccy} = -(a_y - \dot{V}_{Gy}) \qquad (3.34)$$

and

$$A_{ccz} = -(a_z - \dot{V}_{Gz})$$

C. GRID PLATFORM

The condition for the grid platform is described in the section (2-F), specified for mass attraction vertical it gives:

$$w_z = -w_E \sin L_m \qquad (3.35)$$

or

$$\dot{A}_m = \dot{l}_P \sin L_m$$

The components of the angular velocity along x and y axes are, respectively:

$$w_x = \frac{V_{Gy}}{\rho_m} + w_E \cos L_m \cos A_m \qquad (3.36)$$

and

$$w_y = -\frac{V_{Gx}}{\rho_m} - w_E \cos L_m \sin A_m \qquad (3.37)$$

The inertial velocities are expressed by the equations (3.1).

The inertial accelerations are obtained from the general expressions (3.3) to (3.5) specified for grid platform:

$$a_x = \dot{V}_{Gx} + 2\,w_E V_{Gy} \sin L_m - V_{Gz}\left[\frac{V_{Gx}}{\rho_m} + 2\,w_E \cos L_m \sin A_m\right]$$

$$+ \rho_m w_E^2 \sin L_m \cos L_m \cos A_m \qquad (3.38)$$

$$a_y = \dot{V}_{Gy} - 2\, w_E V_{Gx} \sin L_m - V_{Gz} \left[\frac{V_{Gy}}{\rho_m} + 2\, w_E \cos L_m \cos A_m\right]$$

$$- \rho_m w_E^2 \sin L_m \cos L_m \sin A_m \qquad (3.39)$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_{Gx}^2 + V_{Gy}^2}{\rho_m} + 2\, w_E V_{Gx} \cos L_m \sin A_m$$

$$+ 2\, w_E V_{Gy} \cos L_m \cos A_m + \rho_m w_E^2 \cos^2 L_m + G \qquad (3.40)$$

IV. GEODETIC VERTICAL

The vertical taken normal to the surface of the reference ellipsoid represents the gravity vector, which is resultant of the mass attraction and the centrifugal force. The choice of the z axis along this vertical gives:

$$g_x = g_y = 0$$

and

$$g_z = g$$

The additional conditions for the geodetic vertical are:

$$L_k = L_P$$

$$\alpha_k = \alpha_P$$

$$A_k = A_P$$

and

$$\rho_k = \rho_P$$

Observing all those conditions the angular velocities hold the relationship:

$$w_x = \frac{V_{Gy}}{\rho_P} + w_E \cos L_P \cos A_P \tag{4.1}$$

$$w_y = - \frac{V_{Gx}}{\rho_P} - w_E \cos L_P \sin A_P \tag{4.2}$$

and

$$w_z = \dot{A}_P - \frac{\tan L_P}{\rho_P} [V_{Gx} \sin A_P + V_{Gy} \cos A_P] - w_E \sin L_P \tag{4.3}$$

The inertial velocities for geotetic vertical are:

$$V_x = V_{Gx} + \rho_P w_E \cos L_P \sin A_P \tag{4.4}$$

$$V_y = V_{Gy} + \rho_P w_E \cos L_P \cos A_P \tag{4.5}$$

and

$$V_z = V_{Gz} = - \dot{\rho}_P \tag{4.6}$$

where

$$V_{Gx} = \dot{L}_P \rho_P \cos A_P + \dot{l}_P \rho_P \cos L_P \sin A_P \tag{4.7}$$

and

$$V_{Gy} = - \dot{L}_P \rho_P \sin A_P + \dot{l}_P \rho_P \cos L_P \cos A_P \tag{4.8}$$

The inertial accelerations expressed in the terms of the ground and the angular velocities take simple form:

$$a_x = \dot{V}_{Gx} - V_{Gy} [w_z - w_E \sin L_P] + V_{Gz} [w_y - w_E \cos L_P \sin A_P] \tag{4.9}$$

$$a_y = \dot{V}_{Gy} + V_{Gx} [w_z - w_E \sin L_P] - V_{Gz} [w_x + w_E \cos L_P \cos A_P] \tag{4.10}$$

and

$$a_z = \dot{V}_{Gz} - V_{Gx} [w_y - w_E \cos L_P \sin A_P]$$

$$+ V_{Gy} [w_x + w_E \cos L_P \cos A_P] + g \tag{4.11}$$

Gyro torquing rates:

$$t_x = - w_x$$

$$t_y = - w_y \tag{4.12}$$

$$t_z = - w_z$$

and accelerometer torquing signals:

$$A_{ccx} = -(a_x - \dot{V}_{Gx})$$

$$A_{ccy} = -(a_y - \dot{V}_{Gy}) \qquad (4.13)$$

$$A_{ccz} = -(a_z - \dot{V}_{Gz})$$

may be applied as indicated.

A. NORTH SLAVED PLATFORM

The x axis is slaved to the north all the time, therefore $A_P = \dot{A}_P = 0$ and the angular rates are:

$$w_x = \frac{V_{EA}}{\rho_P} + w_E \cos L_P \qquad (4.14)$$

$$w_y = -\frac{V_N}{\rho_P} \qquad (4.15)$$

and

$$w_z = -\frac{V_{EA} \tan L_P}{\rho_P} - w_E \sin L_P \qquad (4.16)$$

where $V_N = \dot{L}_P \rho_P$ and $V_{EA} = \dot{\lambda}_P \rho_P \cos L_P$.

Introducing the angular velocities (4.14) to (4.16) into the equations (4.9) to (4.11) we have the accelerations:

$$a_x = \dot{V}_N + \frac{V_{EA}^2 \tan L_P}{\rho_P} + 2\, w_E V_{EA} \sin L_P - \frac{V_N V_{Gz}}{\rho_P} \qquad (4.17)$$

$$a_y = \dot{V}_{EA} - \frac{V_N V_{EA}}{\rho_P} \tan L_P - 2\, V_N w_E \sin L_P - V_{Gz} \left[\frac{V_{EA}}{\rho_P} + 2\, w_E \cos L_P\right] \tag{4.18}$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_N^2 + V_{EA}^2}{\rho_P} + 2\, w_E V_{EA} \cos L_P + g \tag{4.19}$$

B. FREE AZIMUTH PLATFORM

The z axis is not torqued, therefore

$$w_z = 0 \tag{4.20}$$

Other two angular velocities are:

$$w_x = \frac{V_{Gy}}{\rho_P} + w_E \cos L_P \cos A_P \tag{4.21}$$

$$w_y = -\frac{V_{Gx}}{\rho_P} - w_E \cos L_P \sin A_P \tag{4.22}$$

The inertial velocities are given by the equations (4.4) to (4.6).

Substituting the values of the angular rates (4.20) to (4.22) into the equations of the accelerations (4.9) to (4.11) we have:

$$a_x = \dot{V}_{Gx} + V_{Gy} w_E \sin L_P - V_{Gz} \left[\frac{V_{Gx}}{\rho_P} + 2\, w_E \cos L_P \sin A_P\right] \tag{4.23}$$

$$a_y = \dot{V}_{Gy} - V_{Gx} w_E \sin L_P - V_{Gz} \left[\frac{V_{Gy}}{\rho_P} + 2\, w_E \cos L_P \cos A_P\right] \tag{4.24}$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_{Gx}^2 + V_{Gy}^2}{\rho_P} + 2\, V_{Gx} w_E \cos L_P \sin A_P + 2\, V_{Gy} w_E \cos L_P \cos A_P \qquad (4.25)$$

C. GRID PLATFORM

The angular velocity of the platform along z axis is

$$w_z = -\, w_E \sin L_P \qquad (4.26)$$

Along x and y axes, respectively:

$$w_x = \frac{V_{Gy}}{\rho_P} + w_E \cos L_P \cos A_P \qquad (4.27)$$

$$w_y = -\frac{V_{Gx}}{\rho_P} - w_E \cos L_P \sin A_P \qquad (4.28)$$

The equations (4.4) to (4.6) define the inertial velocities.

The inertial accelerations become:

$$a_x = \dot{V}_{Gx} + 2\, V_{Gy} w_E \sin L_P - V_{Gz} \left[\frac{V_{Gx}}{\rho_P} + 2\, w_E \cos L_P \sin A_P\right] \qquad (4.29)$$

$$a_y = \dot{V}_{Gy} - 2\, V_{Gx} w_E \sin L_P - V_{Gz} \left[\frac{V_{Gy}}{\rho_P} + 2\, w_E \cos L_P \cos A_P\right] \qquad (4.30)$$

and

$$a_z = \dot{V}_{Gz} + \frac{V_{Gx}^2 + V_{Gy}^2}{\rho_P} + 2\ V_{Gx} w_E \cos L_P \sin A_P$$

$$+ 2\ V_{Gy} w_E \cos L_P \cos A_P + g \tag{4.31}$$

V. CONCLUSIONS

The combination of free azimuth and geodetic vertical offers the simplest form of the mathematical equations, since it is closely connected with the actual altitude and ground velocities of the platform and only two platform torquing rates are required.

The types of inertial platforms described in this paper are adequate for moderately high latitudes, but they lead to inaccuracies when applied to operations over the polar areas. The positional and velocity inaccuracies in this case are associated with the rapid convergence of the meridians in polar areas. Such navigational difficulties can be corrected mathematically by use of relocated pole equations, which will be presented in a later paper.

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Report Number TR-31 1 February 1961

INERTIAL PLATFORM EQUATIONS (U)

BUWEPS TASK ASSIGNMENT NO. RAV32F022/3111/F002-13 003

Prepared by:

Peteris Prizevoits
Mathematical Analysis Branch
Theoretical Research Division
Applied Research Department

15 1962

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Page 6 D. Angular Velocities (third line)

$\overline{w}_k$ should be $\overline{w}_x$

Third line from the bottom

$\overline{w}_{Pk}$ should be $\overline{w}_{Px}$

Page 8 Equation (1.13) should be

$$\overline{i} = \overline{I}e_1 + \overline{J}e_2 + \overline{K}e_3$$

$$\overline{j} = \overline{I}m_1 + \overline{J}m_2 + \overline{K}m_3$$

$$\overline{k} = \overline{I}n_1 + \overline{J}n_2 + \overline{K}n_3$$

Equation (1.14) should be

$$\dot{\overline{i}} = \overline{I}\dot{e}_1 + \overline{J}\dot{e}_2 + \overline{K}\dot{e}_3$$

$$\dot{\overline{j}} = \overline{I}\dot{m}_1 + \overline{J}\dot{m}_2 + \overline{K}\dot{m}_3$$

$$\dot{\overline{k}} = \overline{I}\dot{n}_1 + \overline{J}\dot{n}_2 + \overline{K}\dot{n}_3$$

Page 19 (Seventh line from top)

'platform A_k and bearing angle B_x' (should read) bearing angle B_k .

Page 27 C. GRID PLATFORM (Third line) in parallel planes (should read) parallel and displaced .

Page 41 (First line) Change "geotetic" to geodetic .

Page 47 References

e. W.A. Keiskanen (should read) W.A. Heiskanen .

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